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A FRAMEWORK TO FIND THE UPPER BOUND ON POWER OUTPUT AS A FUNCTION OF INPUT VIBRATION PARAMETERS

John Heit **Shad Roundy**

Integrated Self-Powered Sensing Lab, Dept. of Mechanical Engineering
University of Utah, Salt Lake City, UT, 84112, USA

ABSTRACT

This paper outlines a mathematical framework necessary to determine the optimal transducer force for a given vibration input. This relationship, between input vibration parameters and transducer force gives a critical first step in determining the optimal transducer architecture for a given vibration input. This relationship also yields a theoretical maximum energy output for a system with a given proof mass and parasitic mechanical losses, modeled as linear viscous damping.

This relationship is then applied to three specific vibration inputs; a single sinusoid, the sum of two sinusoids, and a single sinusoid with a time dependent frequency (chirp). For the single sinusoidal case, the optimal transducer is found to be a linear spring, resonant with the input frequency, and a linear viscous damper, with matched impedance to the mechanical damping. The resulting transducer force for the input as a sum of two sinusoids is found to be inherently time dependent. This time dependency shows that an active system (not only dependent on the states of the system) can outperform a passive system (dependent only on the states). The final application, for a swept sinusoidal input, results in a transducer of a linear viscous damper, with matched impedance to the mechanical damping, as well as a linear spring with a time dependent coefficient.

INTRODUCTION

Recent work in vibration energy harvesting has focused on ways to improve power output from vibration sources that are not modeled as a single sinusoidal input. Much of this work has investigated the use of nonlinearities as a way to increase energy output [1] [2] [3]. These nonlinearities are usually of the form of a nonlinear spring, such as a Duffing oscillator.

Daqaq *et al.* showed that for Gaussian white noise the energy generation was not a function of the transducer's potential function. That is to say, that the restoring force of the system does not affect the power generation for a Gaussian white noise vibration input. When Daqaq examined the case for filtered white noise, where some frequencies are more represented than others, he was forced to assume a form for the potential function in order to estimate a solution [4].

Hoffmann *et al.* showed that for certain vibration inputs a nonlinear mono-stable or bi-stable oscillator could greatly outperform a linear system. However this work had to assume a form for the restoring force before the parameters could be optimized for power generation. Select results from this study are shown in Table 1 [5].

	Stepped Input	Swept Input	Three Inputs	Bounded White Noise
Linear	100%	100%	100%	100%
Nonlinear Monostable	0%	+479%	0%	+52%
Nonlinear Bistable	+11%	+364%	+11%	+33%

Table 1. Select results from Hoffman *et al* [5].

These example works, and others, give useful insight to the potential uses of nonlinearities for harvesting from complex vibration inputs. However these works do not give a clear relationship between the parameters that define the input vibration and the transducer.

By using methods from the Calculus of Variations this work will find the unconstrained and globally optimal relationship between the input vibration, and force that must be produced by the transducer. This relationship will also define an upper limit for power generated for a given vibration input. This framework will then be applied to three case studies, a single sinusoid, the sum of two sinusoids, and a swept sinusoidal input.

MODELING

A simple, generic model for an inertial energy harvester, as shown in Figure 1, is a kinetic harvester with a generic transducer force F_T that acts on the proof mass. This generic transducer may contain both energy dissipative elements for power generation as well as energy conservative restoring elements. In general, the system is subject to a forcing function $F(t)$. The inherent mechanical losses that are found in any real system are approximated by a linear viscous damper described by a single coefficient b_m. This single degree of freedom system is characterized by a single displacement x. If the system is excited through base excitation, as is the normal case for an inertial generator, then $F(t)$ would be the mass (m) multiplied by the base acceleration $A(t)$. In this case the displacement x is the relative distance between the proof mass and ground. This system is modeled by equation 1.

$$m\ddot{x} + b_m\dot{x} + F_T = F(t) \tag{1}$$

The second order differential equation 1 that models this generic system can be expressed in state space form by letting $x_1 = x, and\ x_2 = \dot{x}$:

$$\dot{x}_1 = x_2 \tag{2}$$

$$\dot{x}_2 = \frac{1}{m}\left(-b_m x_2 - F_T + F(t)\right) \tag{3}$$

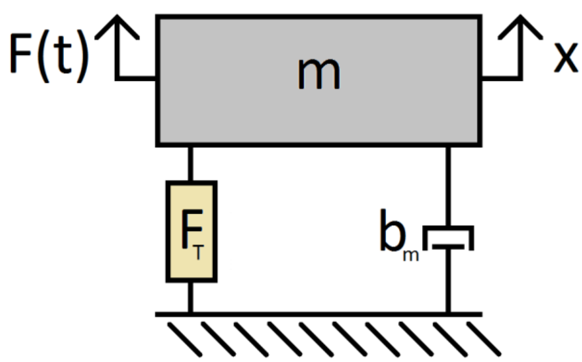

Figure 1. A generic inertial generator characterized by a single displacement x. Here F_T represents the force produced by an unknown transducer architecture. b_m is the coefficient that characterizes the system's linear viscous damping due to inherent mechanical losses of the system.

An energy balance of the system is used in order to find an expression for the energy generated by the transducer as a function of the input. By examining the energy balance of the system in steady state we can neglect the kinetic energy of the mass as well as the possible potential energy stored in the transducer. This is due to the fact that these energy storage elements are restorative, thus they do not represent a net energy input or output to the system while it is in steady state. The energy balance equations are:

$$E_{in} = E_{out} \tag{4}$$

$$\mathrm{E_{in}} = \int \mathrm{F(t)x_2}\ \mathrm{dt} \tag{5}$$

$$\mathrm{E_{out}} = \int \mathrm{b_m x_2^2}\ \mathrm{dt} + \mathrm{E_{gen}} \tag{6}$$

Substituting equation 5 and 6 into 4 will yield an expression for the energy generated as a function of the input force and the velocity of the proof mass.

$$E_{gen} = \int [F(t)x_2 - b_m x_2^2]\ dt \tag{7}$$

For more generalized results we can look at the square of the power to examine a continuous positive definite functional, thereby allowing us to find the critical points in the magnitude of the energy generated.

$$J = \int [F(t)x_2 - b_m x_2^2]^2 dt \tag{8}$$

Equation 8 now represents the energy generated by the transducer as a positive definite functional. If the velocity of the proof mass x_2 is treated as the control parameter, the critical points of the functional can be found through the stationary condition of the Euler-Lagrange equation [6]. Taking I to be the integrand of equation 8 we have:

$$I = [F(t)x_2 - b_m x_2^2]^2 = \\ F(t)^2 x_2^2 - 2b_m F(t) x_2^3 + b_m^2 x_2^4 \tag{9}$$

and the stationary condition to be:

$$\frac{\partial I}{\partial x_2} = 0 \tag{10}$$

In this case the stationary condition yields the critical points of the energy generated with respect to the velocity path of the proof mass. Equation 3 can be used to relate the velocity of the proof mass and the force of the transducer, F_T, acting on the proof mass. This relationship will allow an expression for the necessary transducer force such that the proof mass will follow the calculated optimal velocity path for energy generation. Solving the stationary condition for the critical velocities of x_2:

$$\frac{dI}{dx_2} = 2F(t)^2 x_2 - 6b_m F(t) x_2^2 + 4b_m^2 x_2^3 = 0 \tag{11}$$

$$(F(t) - 2b_m x_2)(F(t) - b_m x_2)x_2 = 0 \tag{12}$$

By factorization, the resulting three solutions are apparent. Here ⋆ denotes a critical path with respect to the energy generated.

$$x_2^\star = \frac{F(t)}{2b_m} \tag{13}$$

$$x_2^\star = \frac{F(t)}{b_m} \tag{14}$$

$$x_2^\star = 0 \tag{15}$$

These three relationships for $x_2^\star$ represent the critical velocity paths, given a vibration input $F(t)$ to the system that will result in a minimum or maximum energy output. By substituting these signals into the second derivative the type of critical points are determined. The second derivative is found to be:

$$\frac{d^2I}{dx^2} = 2F(t)^2 - 12b_mF(t)x_2 + 12b_m^2x_2^2 \tag{16}$$

At $x_2^\star = \frac{F(t)}{2b_m}$:

$$\frac{d^2I}{dx^2} = 2F(t)^2 - 6F(t)^2 + 3F(t)^2 = -F(t)^2 \tag{17}$$

which is negative for all input vibrations $F(t)$.
At $x_2^\star = \frac{F(t)}{b_m}$:

$$\frac{d^2I}{dx^2} = 2F(t)^2 - 12F(t)^2 + 12F(t)^2 = 2F(t)^2 \tag{18}$$

which is positive for all input vibrations $F(t)$.
At $x_2^\star = 0$:

$$\frac{d^2I}{dx^2} = 2F(t)^2 \tag{19}$$

which is positive for all input vibrations $F(t)$.

From this examination we can conclude that $x_2^\star = \frac{F(t)}{2b_m}$ corresponds to the maximum energy generated by the transducer for a given input force. While $x_2^\star = \frac{F(t)}{b_m}$ and $x_2^\star = 0$ correspond to a minimum amount of energy generated.

By substituting these relationships into the governing differential equations 2 – 3, an expression for the displacement of the proof mass x_1 as well as the transducer force F_T can be expressed as a function of the system properties and the input force.
For $x_2^\star = \frac{F(t)}{2b_m}$:

$$x_1^\star = \int \frac{F(t)}{2b_m} dt \tag{20}$$

$$F_T^\star = -\frac{m\dot{F}(t)}{2b_m} + \frac{F(t)}{2} \tag{21}$$

Similarly for $x_2^\star = \frac{F(t)}{b_m}$:

$$x_1^\star = \int \frac{F(t)}{b_m} dt \tag{22}$$

$$F_T^\star = \frac{-m\dot{F}(t)}{b_m} \tag{23}$$

And for $x_2^\star = 0$

$$x_1^\star = 0 \tag{24}$$

$$F_T^\star = F(t) \tag{25}$$

Here, the transducer force, F_T, is an explicit function of time. The optimal transducer can then only be represented as a function of states if the input and its derivative can be expressed as a function of the states through the corresponding relationships of x_1 and x_2. These results are summarized in Table 2. Equations 24 and 25 correspond to the trivial solution of a stationary condition of the proof mass.

SINGLE SINUSOIDAL INPUT

It is difficult to see the relevance of equations 20 – 23 in their general form. To help illustrate these relationships a simple example of a single frequency sinusoidal input will be examined. First we will look at the relationship that maximizes the energy output of the system, and then examine the conditions that create minimum energy output.

For the maximum power condition $x_2^\star = \frac{F(t)}{2b_m}$, letting $F(t) = A\,m\,sin(\omega t)$ results in the following relationships:

$$x_1^\star = -\frac{A\,m}{2b_m\omega}\cos(\omega t) \tag{26}$$

$$x_2^\star = \frac{A\,m}{2b_m}\sin(\omega t) \tag{27}$$

$$F_T{}^\star = -\frac{A\,\omega\,m^2}{2b_m}\cos(\omega t) + \frac{A\,m}{2}\sin(\omega t) \tag{28}$$

Substituting for $x_1^\star$ and $x_2^\star$:

$$F_T^\star = \omega^2 m\,x_1 + b_m x_2 \tag{29}$$

Similarly for $x_2^\star = \frac{A\,m}{b_m}\sin(\omega t)$:

$$F_T^\star = \omega^2 m\,x_1 \tag{30}$$

And for $x_2^\star = 0$:

$$F_T^\star = F(t) \tag{31}$$

Critical Velocity Path	Critical Position Path	Critical Transducer Force	Type
$\mathbf{x_2^\star = \frac{F(t)}{2b_m}}$	$x_1^\star = \int \frac{F(t)}{2b_m} dt$	$F_T^\star = -\frac{m\dot{F}(t)}{2b_m} + \frac{F(t)}{2}$	Maximum
$\mathbf{x_2^\star = \frac{F(t)}{b_m}}$	$x_1^\star = \int \frac{F(t)}{b_m} dt$	$F_T^\star = -\frac{m\dot{F}(t)}{b_m}$	Minimum
$\mathbf{x_2^\star = 0}$	$x_1^\star = 0$	$F_T^\star = F(t)$	Minimum

Table 2. Summary of critical path relationships for a generic input.

The three results for the critical transduction force can be interpreted as follows; for the maximizing condition, $x_2^\star = \frac{F(t)}{2b_m}$, the optimal transducer model is a linear spring and a linear viscous damper for an electrical transducer. The constant of the linear spring is found to be resonant with the vibration input and the impedance of the electrical damper is found to be matched to the impedance of the mechanical damper. That is, b_m, from equations 1 and 3, is the same value as b_m for the transducer in equation 29. This system is a completely passive system, being only a function of the system's states. The system is shown in Figure 2 and modeled by the differential equation 32

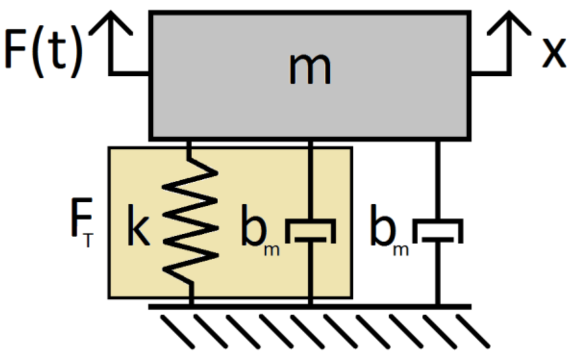

Figure 2. A diagram of the optimal power transducer architecture for an input $F(t) = A\, m\, sin(\omega t)$, where $k = \omega^2 m$.

$$m\ddot{x} + 2b_m\dot{x} + kx = F(t) \tag{32}$$

If the form of the transducer force is assumed to be a linear viscous damper in parallel with a linear spring, the coefficients that result in maximum power generation are widely known and have been previously reported [7]. This framework yielded this previously assumed optimal transducer, without any assumptions on the form of the transducer. By using this simple vibration input, with a partially known solution, this mathematical framework was able to be verified.

For the critical path $x_2^\star = \frac{A\, m}{b_m} \sin(\omega t)$, which corresponds to a minimum power condition the corresponding transducer force is a linear spring with spring constant $k = \omega^2 m$. This system is shown in Figure 3 and modeled by equation 33. While this system has a large response in x_1 to the input $F(t) = A\, m\, sin(\omega t)$, no energy will be converted to useful electric energy since the system's only dissipative element is from parasitic losses due to mechanical damping. The final relationship $x_2^\star = 0$ gives the trivial stationary solution for the energy output, as previously mentioned. Under this condition, the transducer force F_T provides an equal and opposite force to the input $F(t)$ on the mass in order to keep the proof mass stationary. In the case of base excitation, the relative displacement between the proof mass and ground is zero. Physically this would be accomplished by the use of an infinitely stiff spring for the transducer, $k = \infty$.

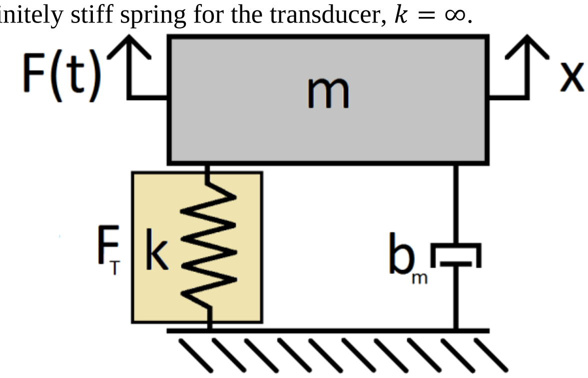

Figure 3. This diagram represents the transduction model for the minimum energy outputs for $x_2^\star = \frac{F(t)}{b_m}$ is $k = \omega^2 m$ and for $x_2^\star = 0$ is $k = \infty$.

$$m\ddot{x} + b_m\dot{x} + kx = F(t) \tag{33}$$

MULTIPLE SINUSOIDAL INPUT

A common vibration input is one of two simultaneous sinusoids at different frequencies. This type of vibration occurs in rotating machinery where two unbalanced masses rotate at different rates fixed relative to one another or in a system where multiple harmonics are well represented.

The RMS power output scales with A^2 for the standard linear system. Thus, the case where the amplitudes of the two sinusoids are equal will be examined. In the case where one sinusoid has an amplitude much greater than the other, it is reasonable to assume that the maximum power generation will be achieved by creating a linear harvester tuned to the ω corresponding to the maximum value of $\frac{A^2}{\omega}$. In the case where the two sinusoids are of similar, but different amplitudes, the following analysis is relevant. The expression for this double sinusoidal input is shown in equation 34.

$$F(t) = A\, m\,(\sin(\omega t) + \sin(n\omega t)) \tag{34}$$

Here, $n \in (0\ \infty)$ represents the multiple difference between the two frequency components.

Examining now only the input-velocity relationship from equation 13 which results in the maximum energy output, the optimal velocity signal for an input of two sinusoids is obtained:

$$x_2^\star = \frac{A\, m}{2b_m}(\sin(\omega t) + \sin(n\omega t)) \tag{35}$$

Using equations 20 and 21 the relationships for the optimal position path and corresponding transducer force to achieve the velocity response as shown in equation 35 can be written as:

$$x_1^\star = -\frac{A\, m}{2\omega b_m}\left(\cos(\omega t) + \frac{1}{n}\cos(n\omega t)\right) \tag{36}$$

$$F_T^\star = \frac{A\, m}{2}\left(sin(\omega t) + sin(n\omega t)\right) - \frac{A\,\omega\, m^2}{2b_m}\left(cos(\omega t) + ncos(n\omega t)\right) \tag{37}$$

Substituting equations 36 and 35 for x_1 and x_2 into equation 37, where available, yields:

$$F_T^\star = b_m x_2 + \omega^2 m\, x_1 + TD \tag{38}$$

Where TD is the time dependent component of the transducer force that cannot be directly substituted for by the systems states x_1 and x_2.

$$TD = \frac{A\,\omega\, m^2}{2b_m}\left(\frac{1}{n} - n\right)\cos(n\omega t) \tag{39}$$

From equation 39 it can be seen that when there is only one frequency component, when $n = 1$, the amplitude of the time dependent portion of the transducer is zero. This intuitive result for the transducer force shows that as $n \rightarrow 1$ the amplitude of the time dependent component goes to zero and the transducer architecture converges to the linear harvester as seen for the single sinusoidal input in equation 29. However, as $n \rightarrow 0\ or\ \infty$ the amplitude of the time dependent portion of the transducer force grows without bound, as is shown in Figure 4.

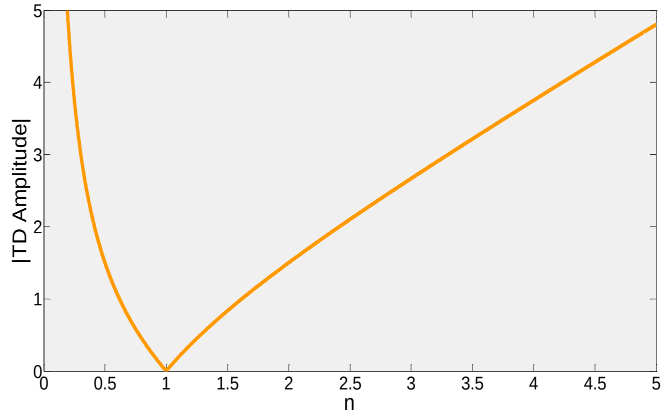

Figure 4. The effect of n on the amplitude of the time dependent component of the transducer force.

The time dependent component of the transducer force shows that the true unconstrained optimal transducer force for an input vibration of this form cannot be realized with a passive system; a system that is only a function of the states. This is due to the complex behavior of the optimal transducer force. In a single period of the proof mass, a different value of the optimal transducer force is required for the same values of the states. This result shows that in principle an active system can outperform a passive system of any type, linear or non-linear. However, this would assume that the restoring force implemented is conservative. An example of this restoring force is shown in Figure 5.

An energy balance is used to determine the nature of the time dependent force. It must be determined if the force does work adding energy to the system over time, takes energy from the system, acting as a complex damping element, or does no net work on the system and acts as a conservative element, such as a spring. The net energy into the proof mass from the time dependent force can be calculated by integrating the force over the displacement for a period T of the entire signal.

$$E_{TD} = \int_0^T TD\; dx_1 = \int_0^T TD * x_2^\star\, dt \tag{40}$$

Substituting equation 39 and 35 for the time dependent portion of the transducer and optimal velocity path respectively.

$$E_{TD}(t) = \frac{A^2 \omega m^3}{4b_m^2}\left(\frac{1}{n} - n\right) * \int_0^T (\sin(\omega t)\cos(n\omega t) + \sin(n\omega t)\cos(n\omega t))dt \tag{41}$$

where T is the complete period of the input vibration. This period can be found by finding the point in time in which the periods of the two sinusoidal components simultaneously occur. This can be found by finding the integer value κ such that $\kappa * n \in \mathbb{Z}^+$. The total period for any input of this form is then defined by $T = \frac{2\pi}{\omega}\kappa$.

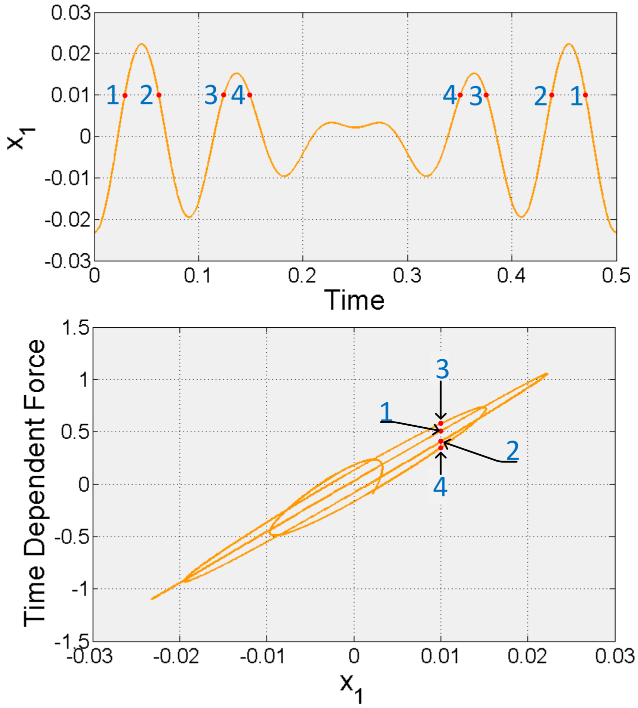

Figure 5. (Top) The steady state position of the proof mass versus time for $n = 1.2$. This complex path repeats itself every period T of the input signal. (Bottom) The time dependent force plotted over the optimal path x_1 over a period T. It can be seen that during one period the same position is repeated multiple times, but requires a different transducer force.

Evaluating equation 41:

$$E_{TD}\left(\frac{2\pi}{\omega}\kappa\right) = \frac{A^2 m^3 \left(\begin{array}{c} 1-(-2+n)n-n(1+n)\mathrm{Cos}[2\kappa(-1+n)\pi]+ \\ (-1+n^2)\mathrm{Cos}[2\kappa n\pi]^2+(-1+n)n\mathrm{Cos}[2\kappa(1+n)\pi] \end{array} \right)}{8b^2n^2} \quad (42)$$

For the constraints of $\kappa \in \mathbb{Z}$ and $n * \kappa \in \mathbb{Z}$ equation 42 reduces to zero. This shows that the time dependent force acts as a conservative element, not doing any work to the system over time.

The upper limit for energy output from the optimal transducer can be shown analytically. This can be accomplished in a similar manner to the derivation of the average power output for the single sinusoid case. Knowing that from the result of equation 38 the power output from the transducer is dissipated by the force of a linear viscous damper, the instantaneous power dissipated through this element can be written as:

$$P = F * v = b_m {x_2^{\star}}^2 \quad (43)$$

Here $x_2^{\star}$ is the optimal velocity shown in equation 35. Integrating the instantaneous power output over time yields the total energy generated by the transducer. Again the upper limit of the integral is defined as the period of the input.

$$E_n = \int_0^{\frac{2\pi}{\omega}\kappa} b_m {x_2^{\star}}^2 dt \quad (44)$$

The integral is then evaluated in the general case for all $n \in (1, \infty)$ as well as the linear case at $n = 1$.

$$E_n = \frac{A^2 m^2 \pi\kappa}{2b_m\omega} \quad (45)$$

$$E_1 = \frac{A^2 m^2 \pi\kappa}{b_m\omega} \quad (46)$$

Examining the ratio $\frac{E_n}{E_1}$ will yield the percentage of energy generated by the optimal transducer for any value of n as compared to the linear system with a single sinusoidal vibration input at $n = 1$.

$$\nu = \frac{E_n}{E_1} = \frac{1}{2} \quad (47)$$

That is to say, the maximum amount of energy that can be extracted from an input of two sinusoids separated in frequency by a factor n is half of the energy that can be produced by a linear system under a single sinusoid input of twice the amplitude. Another useful comparison of the energy output can be made with a linear system harvesting from only the lower of the two frequencies. In this case, the optimal transducer will produce twice the energy of the linear system.

To verify these results a numeric study was performed. This study was performed using Matlab's ODE45 function. The energy output was measured after the system achieved steady state, in order to avoid transients affecting the solution. The results of this study confirm the analytical derivations above. An output of this study can be seen in Figure 6.

It can be seen that from the linear power equation that the power output from the system is proportional to the amplitude squared over ω, that is, $\propto \frac{A^2}{\omega}$ [7]. In the special case of $n = 1$, $P \propto \frac{(A+A)^2}{\omega} = \frac{4A^2}{\omega}$. However, this simple substitution is only valid for the special case when $n = 1$. For a small perturbation of the upper sinusoid, $n = 1 + \delta$, the power output is $P \propto \frac{A^2}{\omega} + \frac{A^2}{\omega(1+\delta)}$. Where for a sufficiently small δ, $P \propto \frac{2A^2}{\omega}$. This power output for the slightly perturbed case is half of the power output of the case when $n = 1$. As n grows larger the power output becomes $P \propto \frac{A^2}{\omega} + \frac{A^2}{n\omega}$. It can be seen that for large values of n, $P \propto \frac{A^2}{\omega}$, which is a quarter of the power output seen in the case when $n = 1$.

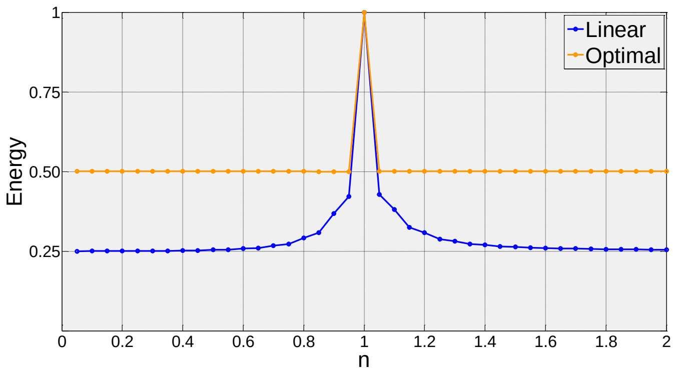

Figure 6. Numeric simulations of the energy output of the optimal transducer as compared to a linear harvester. The energy production has been normalized by the energy output of both systems at $n = 1$.

SWEPT SIUSOIDAL INPUT

Another common vibration input is one of a single sinusoid with a time dependent frequency. A common occurrence of this input type is found in a variety of transportation applications. These applications range from the quickly varying rotational speed of an automobile tire, as experienced by a tire pressure monitor to the slow changing excitation experienced by trains. Machinery with an unbalanced mass, whose rotational speed is time dependent, also experiences this type of excitation such as that found in many industrial and manufacturing applications. Another occurrence of this input is in structural health monitoring. In this application the fundamental frequency of the structure changes very slowly with ambient conditions such as temperature.

Using the relationship from equation 13, corresponding to the maximum power output, the optimal velocity path as a function of the vibration input is described by:

$$x_2^\star = \frac{A\,m}{2b_m} \mathrm{Sin}\left[2\pi\left(\mathrm{f}_0 + \frac{1}{2} f_r t\right) t\right] \tag{48}$$

Equation 3 can be used to find the relationship for the optimal transducer force and is shown in equation 49. However, in this case equation 2 cannot be used to solve for the optimal position path of the proof mass x_1 as an analytical solution to the integral of $x_2^\star$ can only be expressed through the use of Fresnel integrals. This relationship does not allow for a direct substitution of a function of x_1 for the optimal transducer force.

$$F_T^\star = \frac{A\,m}{2} \mathrm{Sin}\left[2\pi\left(\mathrm{f}_0 + \frac{1}{2} f_r t\right) t\right] - \frac{\pi\,A\,m^2}{b_m}(f_0 + f_r t)\mathrm{Cos}\left[2\pi\left(\mathrm{f}_0 + \frac{1}{2} f_r t\right) t\right] \tag{49}$$

The same substitution, as for the single and double sinusoid, can be made for the first term of equation 49 by a linear viscous damper model with matched impedance to the mechanical damping:

$$F_T^\star = b_m x_2 + TD \tag{50}$$

Here, TD is again the time dependent portion of the transducer force that cannot be substituted directly by the states of the system.

$$TD = -\frac{\pi A m^2}{b_m}(f_0 + f_r t)\mathrm{Cos}\left[2\pi\left(\mathrm{f}_0 + \frac{1}{2} f_r t\right) t\right] \tag{51}$$

With the single sinusoid case in mind the mathematical framework can be used to check the validity of the assumed optimal transducer. Specifically, the optimal transducer is assumed to be a linear spring with a time dependent stiffness coefficient that is resonant with the input frequency at all times. Since the viscous damping which represents energy generation is already expressed in equation 50, the missing component of the assumed transducer is this time-varying spring component. It is assumed that the time dependent portion of the transducer force is this time dependent spring. Mathematically this assumption is expressed as:

$$TD = k(t)x_1 = \omega(t)^2 m\, x_1 = \left(2\pi(f_0 + f_r t)\right)^2 m\, x_1 \tag{52}$$

This relationship can be used to find an expression for x_1 that can subsequently be differentiated for x_2. To validate the accuracy of our assumption in equation 52, x_2 from the assumed optimal transducer architecture will be compared to $x_2^\star$ in equation 48.

Substituting equation 51 into equation 52 and solving for x_1 yields:

$$x_1 = -\frac{A\,m}{4\pi b_m(f_0 + f_r t)} \mathrm{Cos}\left[2\pi\left(\mathrm{f}_0 + \frac{1}{2} f_r t\right) t\right] \tag{53}$$

By differentiation the velocity path of the assumed optimal transducer x_2 is found to be:

$$x_2 = \frac{A\,m\,f_r}{4\pi b_m (f_0 + f_r t)^2} \mathrm{Cos}\left[2\pi\left(\mathrm{f}_0 + \frac{1}{2} f_r t\right) t\right] + \frac{A\,m}{2b_m} \mathrm{Sin}\left[2\pi\left(\mathrm{f}_0 + \frac{1}{2} f_r t\right) t\right] \tag{54}$$

Examining the two components of equation 54 we can see that the second term is identical to the expression of $x_2^\star$ in equation 48. The first term of the expression is a transient sinusoid with decaying amplitude. An example comparison between x_2 and $x_2^\star$ is shown in Figure 7 with unitary values of the parameters and a relatively large value for f_r of 10 Hz/sec.

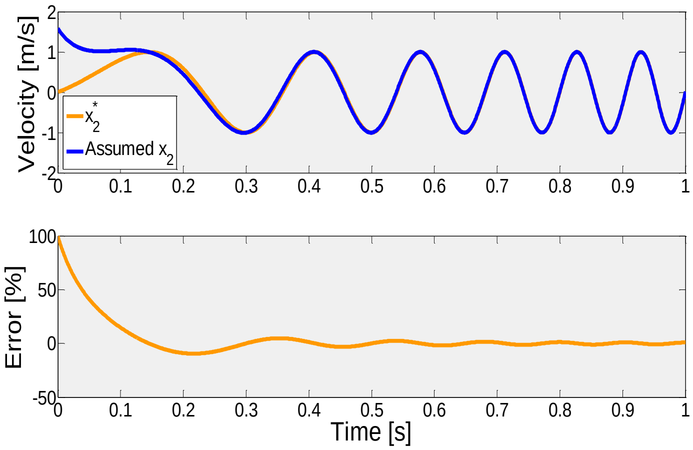

Figure 7. (Top) x_2 and $x_2^\star$ versus time for $A = 1, m = 1, b_m = 1, f_0 = 1, f_r = 10$. (Bottom) The percentage error between the two signals in time. The error quickly decays to zero as the amplitude of the transient decays.

The transient component that causes the incongruity between x_2 and $x_2^\star$ is only appreciable for large values of f_r over small time scales relative to the initial period $\frac{1}{f_0}$. In application, the value of f_r will generally be fairly small as any structure will require an input of energy to adjust the resonance frequency of the system.

Through this example we have shown that the mathematical framework can be used to validate an assumed optimal architecture. This framework can be used as a basis of comparison between transducer architectures and as a mark of feasibility for implementing a vibration energy harvester for a given vibration input and power requirement.

The upper limit for the energy output of the optimal transducer for a swept sinusoidal input can be found in the same manner as the double sinusoid. By looking at the energy balance in equation 4 the energy generated can be described as:

$$E_{gen} = \int_0^t \left[F(t)x_2^\star - b_m {x_2^\star}^2\right] d\tau = \frac{A^2m^2\left(2\sqrt{f_r}t + \mathrm{Cos}\left[\frac{2f_0^2\pi}{f_r}\right]\left(C\left[\frac{2f_0}{\sqrt{fr}}\right] - C\left[\frac{2(f_0+f_r t)}{\sqrt{f_r}}\right]\right) + \left(S\left[\frac{2f_0}{\sqrt{fr}}\right] - S\left[\frac{2(f_0+f_r t)}{\sqrt{f_r}}\right]\right)\mathrm{Sin}\left[\frac{2f_0^2\pi}{f_r}\right]\right)}{16 b_m \sqrt{f_r}} \tag{55}$$

Where $C[\,]$ and $S[\,]$ represent the Fresnel integrals; defined as $C[v] = \int_0^v Cos\left[\frac{\pi t^2}{2}\right] dt$ and $S[v] = \int_0^v Sin\left[\frac{\pi t^2}{2}\right] dt$, respectively. Noting the boundedness of $C[\,]$ and $S[\,]$ an approximation of the energy generated over large periods of time can be expressed as:

$$E_{gen} \cong \frac{A^2m^2}{8b_m} t \tag{56}$$

This approximation converges more quickly in time for large values of the ratio $\frac{2f_0}{\sqrt{fr}}$. That is to say this approximation is more accurate for slow changes in the input frequency relative to the starting frequency. For relatively large time intervals the bounded components of the energy are insignificant compared to the time dependent components.

The average power output can be expressed as:

$$P_{RMS} = \frac{\mathrm{dE_{gen}}}{\mathrm{dt}} = \frac{A^2m^2}{8b_m} \tag{57}$$

This power output is identical to the power output found for a linear system harvesting from a single sinusoid. It is expressed using the damping coefficient, (b_m), rather than the damping ratio, (ζ_m) and therefore, is not an explicit function of frequency.

Conclusion

This paper has outlined a framework necessary to relate the form of an input vibration to an optimal transducer force. In creating this framework no assumptions of the transducer architecture were made. This framework was then applied to three case studies. The first was a vibration input of a single sinusoid. The optimal transducer was found to be a linear viscous damper with matched impedance, and a linear spring, resonant to the input frequency. This solution can be expressed as a function of the systems states so is considered a passive system. While the solution of this case study seems trivial, it accomplishes two objectives. First, to the author's knowledge this is the first time that the optimal form of the transducer force for a sinusoidal input has been explicitly proven. Second, it serves to validate the methodology.

The second application was an input of the sum of two sinusoids at different frequencies. The optimal transducer force found was dependent on the difference between the two frequencies. In all cases the optimal transducer force consists of a linear viscous damper with matched impedance, a linear spring, and a time dependent component. This time dependent component was found to act as a conservative force, like a time dependent spring. The framework was used to find the upper limit for power generation. This limit was found to be twice the power output of a linear system harvesting only from the lower of the two frequency components.

The final application was for a swept sinusoidal input. Here optimal transducer contained two portions, a linear viscous damper and a time dependent component. Here an assumed solution, based on the optimal solution for a stationary sinusoid, was checked against the optimal solution of the framework. It was found that the assumed solution quickly converged to optimal solution.

This basic framework could be applied to vibration inputs of various forms to determine the upper bound of power generation for that type of vibration, and the optimal transducer architecture. If a transducer architecture is assumed, a Duffing oscillator for example, this methodology can be applied to determine how close to the assumed solution is to the upper bound.

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